ON THE GENERAL PERTURBATIONS OF THE POSITION VECTORS OF A PLANETARY SYSTEM

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Abstract 33371

In this article a theory of the general perturbations of a planetary system is developed.

The perturbations of the position vectors of each planet are expanded into series arranged in powers and products of the masses m_1, m_2, ..., m_n of the planets constituting the system. Perturbations of different orders are obtained in the form of series containing the purely periodic, the secular and the mixed terms, in accordance with standard astronomical practice. The influence of the lower order perturbations on the higher order ones is determined.

Typical differential equations are formed determining those perturbations of the i^{th} planet which are proportional to m_k, $m_k m_p$, $m_k m_p m_s$, ... The right sides of these differential equations are obtained as the corresponding terms in the Maxwellian expansion of the gravitational forces in terms of multipoles. The momenta of these multipoles are the perturbations of all possible orders.

The explicit calculation is carried out here for the perturbations of the first, second and third orders, and the procedure for determining the higher order perturbations is outlined.

Decomposing each perturbation of any particular planet m_i along the undisturbed position vector $\vec{r}_i$, along the undisturbed velocity $\vec{v}_i$, and along the unit vector $\vec{R}_i$ normal to the undisturbed orbital plane, we deduce the differential equations to a form easily integrable by quadratures. After the integration it is more convenient in practical applications to replace this decomposition of perturbations by the decomposition along $\vec{r}_i$, $\vec{R}_i \times \vec{r}_i$ and $\vec{R}_i$. The problem of the constants of integration is treated for the case of the elements osculating at the initial moment and for the case of the mean elements. The results given here extend and generalize the author's previous results on the case of the whole planetary system. The method suffers, however, from disadvantages common to all astronomical methods of the general planetary perturbations. It is not applicable to pair of planets if their orbits approach each other very closely.

[redacted]

Basic Notations

f = the gravitational constant

m_i = the mass of the i^{th} planet, the mass of the sun is put equal to one

μ_i^2 = $f(1+m_i)$

$\vec{r}_i$ = the undisturbed position vector of the i^{th} planet

r_i = $|\vec{r}_i|$

$\vec{R}_i$ = the unit vector normal to the undisturbed orbit plane of the i^{th} planet

$\vec{v}_i$ = $\dfrac{d\vec{r}_i}{dt}$

$\delta\vec{r}_i$ = the perturbations in the position vector of the i^{th} planet

$\vec{r}_i^{\,\alpha}$ = perturbations in $\vec{r}_i$ proportional to m_α

$\vec{r}_i^{\,\alpha\beta}$ = perturbations in $\vec{r}_i$, proportional to $m_\alpha m_\beta$

$\vec{r}_i^{\,\alpha\beta\gamma}$ = perturbations in $\vec{r}_i$ proportional to $m_\alpha m_\beta m_\gamma$

$\vec{\rho}_{ki}$ = $\vec{r}_k - \vec{r}_i$

ρ_{ki} = $|\vec{\rho}_{ki}|$

$\delta\vec{\rho}_{ki}$ = $\delta\vec{r}_k - \delta\vec{r}_i$

$\vec{\rho}_{ki}^{\,\alpha}$ = $\vec{r}_k^{\,\alpha} - \vec{r}_i^{\,\alpha}$

$\vec{\rho}_{ki}^{\,\alpha\beta}$ = $\vec{r}_k^{\,\alpha\beta} - \vec{r}_i^{\,\alpha\beta}$

∇_i = the del operator with respect to $\vec{r}_i$

D_i = $\nabla_i \exp(\delta\vec{r}_i \cdot \nabla_i)$

D_{ji} = $\nabla_i \exp(\delta\vec{\rho}_{ji} \cdot \nabla_j)$

= $-\nabla_j \exp(\delta\vec{\rho}_{ji} \cdot \nabla_j)$

$\vec{\rho}_{ki}^{\,\alpha\beta\gamma}$ = $\vec{r}_k^{\,\alpha\beta\gamma} - \vec{r}_i^{\,\alpha\beta\gamma}$

ON THE GENERAL PERTURBATIONS OF THE POSITION VECTORS OF A PLANETARY SYSTEM

INTRODUCTION

In this article a theory of general perturbations of a planetary system is developed. The perturbations in the position vector of each planet is developed into a series in powers and products of the disturbing masses and into a series containing the periodic, the secular, and the mixed terms with respect to time.

Such a way of representing the integrals of the disturbed motion is in accordance with standard astronomical practice. From the purely mathematical standpoint, this solution can be affected by all the difficulties associated with the near resonance conditions caused by the small divisors.

We establish the differential equations for perturbations proportional to the powers and products of masses in a form integrable by quadratures. The explicit calculation is carried out through the perturbations of the third order. In our planetary system it is rarely necessary to include the perturbations of the fourth and higher orders. However, an outline of the procedure for including the perturbations of even higher orders is indicated here.

The problem of a direct determining of the general perturbations in the position vectors, including the effects of higher orders, became possible only in recent years with the advent of electronic computers. By decomposing the perturbations $\delta\vec{r}_i$ along the directions of $\vec{r}_i$, $\vec{v}_i$ and $\vec{R}_i$, [Musen and Carpenter, 1963] one can integrate the variational equation of the problem by Hill's [1874] procedure directly without resorting to the method of variation of astronomical constants. We shall use here the same decomposition as an intermediary step; but the final decomposition of the perturbations will be along $\vec{r}_i$, $\vec{R}_i \times \vec{r}_i$ and $\vec{R}_i$, in order to reduce the components of the disturbing term on the right side of the variational equation to a simple form. In computing the higher order

perturbations, it will be necessary to expand the disturbing forces in powers of perturbations in the position vectors. Maxwell's method of expanding the electrostatic potential in terms of multipoles by employing symbolic operators [1904, 3rd ed] can be used profitably also in planetary theories. In our exposition the moments of the multipoles are the perturbations of different orders of the position vectors. Evidently any other way of expanding the disturbing forces in terms of the perturbations of the position vectors will lead to a duplication of Maxwell's expansion, but through a more laborious writing.

In the theory of perturbations of the position vectors we achieve economy of theoretical thinking as well as economy of computing machine time, because a set of homogeneous operations is being constantly repeated. All these circumstances suggest that future methods of calculating general perturbations will be based on the expansion of the perturbations in the position vectors directly.

The Differential Equations of the Problems

Putting

$$\vec{\rho} = \vec{r} - \vec{a}$$

we shall make use of Maxwell expansion of the spherical functions as defined in terms of multipoles. We have

$$\phi^{(n)} = \left(\prod_{k=1}^{n} \vec{a}_k \cdot \nabla\right) \frac{1}{\rho} = (-1)^n \left[\frac{1.3\cdots(2n-1)}{\rho^{2n+1}} \prod_{k=1}^{n} \vec{a}_k \cdot \vec{\rho}\right. \tag{1}$$

$k = 1, 2, \cdots, n$

$n = 1, 2, 3, \cdots$

$$- \frac{1.3\cdots(2n-3)}{\rho^{2n-1}} \sum \vec{a}_1 \cdot \vec{a}_2 \prod_{k=3}^{n} \vec{a}_k \cdot \vec{\rho}$$

$$\left. + \frac{1.3\cdots(2n-5)}{\rho^{2n-3}} \sum \vec{a}_1 \cdot \vec{a}_2 \; \vec{a}_3 \cdot \vec{a}_4 \prod_{k=5}^{n} \vec{a}_k \cdot \vec{\rho} \cdots \right],$$

here $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are constant vectors, ∇ is the del-operator with respect to $\vec{r}$, and the sums

$$\sum \vec{a}_1 \cdot \vec{a}_2 \prod_{k=3}^{n} \vec{a}_k \cdot \vec{\rho}, \text{ etc.}$$

designate the sums of all terms as obtained from the first term by means of the permutations of all n indices. In particular, we have

$$\phi^{(0)} = \frac{1}{\rho}, \tag{2}$$

$$\phi^{(1)} = \vec{a}_1 \cdot \nabla \frac{1}{\rho} = -\frac{\vec{a}_1 \cdot \vec{\rho}}{\rho^3}, \tag{3}$$

$$\phi^{(2)} = \vec{a}_1 \cdot \nabla \vec{a}_2 \cdot \nabla \frac{1}{\rho} = +\frac{3}{\rho^5} \vec{\rho} \cdot \vec{a}_1 \; \vec{\rho} \cdot \vec{a}_2 - \frac{1}{\rho^3} \vec{a}_1 \cdot \vec{a}_2, \tag{4}$$

$$\phi^{(3)} = \vec{a}_1 \cdot \nabla \vec{a}_2 \cdot \nabla \vec{a}_3 \cdot \nabla \frac{1}{\rho} = \tag{5}$$

$$= -\frac{15}{\rho^7} \vec{\rho} \cdot \vec{a}_1 \; \vec{\rho} \cdot \vec{a}_2 \; \vec{\rho} \cdot \vec{a}_3 + \frac{3}{\rho^5} (\vec{\rho} \cdot \vec{a}_1 \; \vec{a}_2 \cdot \vec{a}_3 + \vec{\rho} \cdot \vec{a}_2 \; \vec{a}_3 \cdot \vec{a}_1$$

$$+ \vec{\rho} \cdot \vec{a}_3 \; \vec{a}_1 \cdot \vec{a}_2),$$

$$\phi^{(4)} = \vec{a}_1 \cdot \nabla \vec{a}_2 \cdot \nabla \vec{a}_3 \cdot \nabla \vec{a}_4 \cdot \nabla \frac{1}{\rho} = \tag{6}$$

$$= + \frac{105}{\rho^9} \vec{\rho}\cdot\vec{a}_1 \; \vec{\rho}\cdot\vec{a}_2 \; \vec{\rho}\cdot\vec{a}_3 \; \vec{\rho}\cdot\vec{a}_4$$

$$- \frac{15}{\rho^7} (\vec{\rho}\cdot\vec{a}_2 \; \vec{\rho}\cdot\vec{a}_3 \; \vec{a}_1\cdot\vec{a}_4 + \vec{\rho}\cdot\vec{a}_3 \; \vec{\rho}\cdot\vec{a}_1 \; \vec{a}_2\cdot\vec{a}_4$$

$$+ \vec{\rho}\cdot\vec{a}_1 \; \vec{\rho}\cdot\vec{a}_2 \; \vec{a}_3\cdot\vec{a}_4 + \vec{\rho}\cdot\vec{a}_1 \; \vec{\rho}\cdot\vec{a}_4 \; \vec{a}_2\cdot\vec{a}_3$$

$$+ \vec{\rho}\cdot\vec{a}_2 \; \vec{\rho}\cdot\vec{a}_4 \; \vec{a}_1\cdot\vec{a}_3 + \vec{\rho}\cdot\vec{a}_3 \; \vec{\rho}\cdot\vec{a}_4 \; \vec{a}_1\cdot a_2)$$

$$+ \frac{3}{\rho^5} (\vec{a}_1\cdot\vec{a}_4 \; \vec{a}_2\cdot\vec{a}_3 + \vec{a}_2\cdot\vec{a}_4 \; \vec{a}_3\cdot\vec{a}_1 + \vec{a}_3\cdot\vec{a}_4 \; \vec{a}_1\cdot\vec{a}_2),$$

. . . .

The gradient of the spherical function $\varphi^{(n)}$ is obtained from the Maxwellian expansion of $\phi^{(n+1)}$ simply by replacing the moment $\vec{a}_{n+1}$ by the idemfactor I. Thus from (2) - (6):

$$\nabla\phi^{(0)} = \nabla\frac{1}{\rho} = -\frac{\vec{\rho}}{\rho^3}, \tag{7}$$

$$\vec{a}_1\cdot\nabla\nabla\frac{1}{\rho} = +\frac{3}{\rho^5}\vec{\rho}\;\vec{\rho}\cdot\vec{a}_1 - \frac{1}{\rho^3}\vec{a}_1, \tag{8}$$

$$\vec{a}_1\cdot\nabla\,\vec{a}_2\cdot\nabla\,\nabla\frac{1}{\rho} = -\frac{15}{\rho^7}\vec{\rho}\;\vec{\rho}\cdot\vec{a}_1\;\vec{\rho}\cdot\vec{a}_2 \tag{9}$$
$$+\frac{3}{\rho^5}(\vec{\rho}\cdot\vec{a}_1\;\vec{a}_2 + \vec{\rho}\cdot\vec{a}_2\;\vec{a}_1 + \vec{\rho}\;\vec{a}_1\cdot\vec{a}_2),$$

$$\vec{a}_1 \cdot \nabla\ \vec{a}_2 \cdot \nabla\ \vec{a}_3 \cdot \nabla\ \nabla \frac{1}{\rho} = + \frac{105}{\rho^7} \vec{\rho}\ \vec{\rho} \cdot \vec{a}_1\ \vec{\rho} \cdot \vec{a}_2\ \vec{\rho} \cdot \vec{a}_3 \tag{10}$$

$$- \frac{15}{\rho^7} (\vec{\rho} \cdot \vec{a}_2\ \vec{\rho} \cdot \vec{a}_3\ \vec{a}_1 + \vec{\rho} \cdot \vec{a}_3\ \vec{\rho} \cdot \vec{a}_1\ \vec{a}_2$$

$$+ \vec{\rho} \cdot \vec{a}_1\ \vec{\rho} \cdot \vec{a}_2\ \vec{a}_3 + \vec{\rho}\ \vec{\rho} \cdot \vec{a}_1\ \vec{a}_2 \cdot \vec{a}_3$$

$$+ \vec{\rho}\ \vec{\rho} \cdot \vec{a}_2\ \vec{a}_1 \cdot \vec{a}_3 + \vec{\rho}\ \vec{\rho} \cdot \vec{a}_3\ \vec{a}_1 \cdot \vec{a}_2)$$

$$+ \frac{3}{\rho^5} (\vec{a}_1\ \vec{a}_2 \cdot \vec{a}_3 + \vec{a}_2\ \vec{a}_3 \cdot \vec{a}_1 + \vec{a}_3\ \vec{a}_1 \cdot \vec{a}_2)$$

. . . .

The differential equation of the disturbed motion of the i^{th} planet can be written in the form

$$\frac{d^2}{dt^2} (\vec{r}_i + \delta \vec{r}_i) = \nabla_i \frac{\mu_i^2}{|\vec{r}_i + \delta \vec{r}_i|} + \sum_{\substack{\sigma=1 \\ \sigma \neq i}}^{n} f m_\sigma \nabla_\sigma \left(- \frac{1}{|\vec{\rho}_{\sigma i} + \delta \vec{\rho}_{\sigma i}|} + \frac{1}{|\vec{r}_\sigma + \delta \vec{r}_\sigma|} \right) \tag{11}$$

$i,\ \sigma = 1, 2, \cdots n.$

Taking the equation

$$\frac{d^2 \vec{r}_i}{dt^2} = \nabla_i \frac{\mu_i^2}{r_i}$$

into account and introducing the vectorial differential operators

$$D_i = \nabla_i \exp(\delta \vec{r}_i \cdot \nabla_i), \tag{12}$$

$$D_{ji} = \nabla_j \exp(\delta \vec{\rho}_{ji} \cdot \nabla_j), \tag{13}$$

which perform the development of (11) into a power series in the components of $\delta \vec{r}_k$ $(k = 1, 2, \cdots n)$, we obtain from (11) the differential equation for $\delta \vec{r}_i$ in the form

$$\frac{d^2 \delta \vec{r}_i}{dt^2} = \mu_i^2 (D_i - \nabla_i) \frac{1}{r_i} + \sum_{\substack{\sigma=1 \\ \sigma \neq i}}^{n} f m_\sigma \left(- D_{\sigma i} \frac{1}{\rho_{\sigma i}} + D_\sigma \frac{1}{r_\sigma} \right). \tag{14}$$

The perturbation vector $\delta \vec{r}_i$ can be developed into a series with respect to the powers and products of the disturbing masses. We put

$$\delta \vec{r}_i = \frac{1}{1!} \sum_{\alpha} \vec{r}_i^{\alpha} + \frac{1}{2!} \sum_{\alpha\beta} \vec{r}_i^{\alpha\beta} + \frac{1}{3!} \sum_{\alpha\beta\gamma} \vec{r}_i^{\alpha\beta\gamma} + \cdots, \tag{15}$$

where $\vec{r}_i^{\alpha}$ is proportional to m_α, and $\vec{r}_i^{\alpha\beta}$ is proportional to $m_\alpha m_\beta$, etc. The factors in front of the sums in (15) are introduced in order to remove large coefficients in higher approximations.

We define $\vec{r}_i^{\alpha\beta}$, $\vec{r}_i^{\alpha\beta\gamma}$, ... in such a way that they remain invariant under the permutations of the upper indices:

$$\vec{r}_i^{\alpha\beta} = \vec{r}_i^{\beta\alpha}, \quad \vec{r}_i^{\alpha\beta\gamma} = \vec{r}^{\beta\gamma\alpha} = \cdots, \text{ etc.}$$

Symbolically,

$$\delta \vec{r}_i = \left(\exp \sum_{\alpha} \vec{r}_i^{\alpha} \right) - 1,$$

where in performing the development and the symbolic "multiplications" the indices are not being added, but written in a row. We have also

$$\vec{r}_i^{\,i} = \vec{r}_i^{\,ii} = \vec{r}_i^{\,iii} = \cdots = 0.$$

We shall now deduce the differential equation for determining perturbations of the form

$$\vec{r}_i^{\,k}, \quad \vec{r}_i^{\,kp}, \ \vec{r}_i^{\,kpq}, \ \vec{r}_i^{\,kpqs}, \cdots, \qquad (i, k, p, \cdots = 1, 2, \cdots n),$$

first under the assumption that there are no identical indices among k, p, q,... Retaining only the substantial terms, we have

$$\begin{aligned} \delta\vec{r}_i &= (\vec{r}_i^{\,k} + \vec{r}_i^{\,p} + \vec{r}_i^{\,q} + \vec{r}_i^{\,s}) \\ &+ (\vec{r}_i^{\,kp} + \vec{r}_i^{\,kq} + \vec{r}_i^{\,ks} + \vec{r}_i^{\,pq} + \vec{r}_i^{\,ps} + \vec{r}_i^{\,qs}) \\ &+ (\vec{r}_i^{\,kpq} + \vec{r}_i^{\,kps} + \vec{r}_i^{\,kqs} + \vec{r}_i^{\,pqs}) \\ &+ \vec{r}_i^{\,kpqs} + \cdots, \end{aligned} \tag{16}$$

$$\begin{aligned} \delta\vec{\rho}_{ji} &= (\vec{\rho}_{ji}^{\,k} + \vec{\rho}_{ji}^{\,p} + \vec{\rho}_{ji}^{\,q}) \\ &+ (\vec{\rho}_{ji}^{\,kp} + \vec{\rho}_{ji}^{\,kq} + \vec{\rho}_{ji}^{\,pq}) \\ &+ \vec{\rho}_{ji}^{\,kpq} + \cdots . \end{aligned} \tag{17}$$

Substituting these values into

$$D_i = \nabla_i + \frac{\delta\vec{r}_i \cdot \nabla_i}{1!} \nabla_i + \frac{(\delta\vec{r}_i \cdot \nabla_i)^2}{2!} \nabla_i + \cdots,$$

$$D_{ji} = \nabla_j + \frac{\delta\vec{\rho}_{ji}\cdot\nabla_j}{1!}\nabla_j + \frac{(\delta\vec{\rho}_{ji}\cdot\nabla_j)^2}{2!}\nabla_j + \cdots$$

and again retaining only the necessary terms, we deduce:

$$\begin{aligned}
D_i - \nabla_i = {} & \vec{r}_i^{\,k}\cdot\nabla_i\,\nabla_i + (\vec{r}_i^{\,kp}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,k}\cdot\nabla_i\,\vec{r}_i^{\,p}\cdot\nabla_i\,\nabla_i) \\
& + (\vec{r}_i^{\,kpq}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,kp}\cdot\nabla_i\,\vec{r}_i^{\,q}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,pq}\cdot\nabla_i\,\vec{r}_i^{\,k}\cdot\nabla_i\,\nabla_i \\
& + \vec{r}_i^{\,qk}\cdot\nabla_i\,\vec{r}_i^{\,p}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,k}\cdot\nabla_i\,\vec{r}_i^{\,p}\cdot\nabla_i\,\vec{r}_i^{\,q}\cdot\nabla_i\,\nabla_i) \\
& + \Big[\vec{r}_i^{\,kpqs}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,kpq}\cdot\nabla_i\,\vec{r}_i^{\,s}\cdot\nabla_i\,\nabla_i \\
& + \vec{r}_i^{\,kps}\cdot\nabla_i\,\vec{r}_i^{\,q}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,kqs}\cdot\nabla_i\,\vec{r}_i^{\,p}\cdot\nabla_i\,\nabla_i \\
& + \vec{r}_i^{\,pqs}\cdot\nabla_i\,\vec{r}_i^{\,k}\cdot\nabla_i\,\nabla_i + (\vec{r}_i^{\,kp}\cdot\nabla_i\,\vec{r}_i^{\,qs}\cdot\nabla_i\,\nabla_i \\
& + \vec{r}_i^{\,kq}\cdot\nabla_i\,\vec{r}_i^{\,ps}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,ks}\cdot\nabla_i\,\vec{r}_i^{\,pq}\cdot\nabla_i\,\nabla_i) \\
& + \frac{1}{2}(\vec{r}_i^{\,kp}\cdot\nabla_i\,\vec{r}_i^{\,q}\cdot\nabla_i\,\vec{r}_i^{\,s}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,kq}\cdot\nabla_i\,r_i^{s}\cdot\nabla\,\vec{r}_i^{\,p}\cdot\nabla_i\,\nabla_i \\
& + \vec{r}_i^{\,ks}\cdot\nabla_i\,\vec{r}_i^{\,p}\cdot\nabla_i\,\vec{r}_i^{\,q}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,pq}\cdot\nabla_i\,\vec{r}_i^{\,k}\cdot\nabla_i\,\vec{r}_i^{\,s}\cdot\nabla_i\,\nabla_i \\
& + \vec{r}_i^{\,ps}\cdot\nabla_i\,\vec{r}_i^{\,k}\cdot\nabla_i\,\vec{r}_i^{\,q}\cdot\nabla_i\,\nabla_i + \vec{r}_i^{\,qs}\cdot\nabla_i\,\vec{r}_i^{\,k}\cdot\nabla_i\,\vec{r}_i^{\,p}\cdot\nabla_i\,\nabla_i) \\
& + \vec{r}_i^{\,k}\cdot\nabla_i\,\vec{r}_i^{\,p}\cdot\nabla_i\,\vec{r}_i^{\,q}\cdot\nabla_i\,\vec{r}_i^{\,s}\cdot\nabla_i\,\nabla_i\Big] + \cdots
\end{aligned} \tag{18}$$

$$D_{ji} = \nabla_j + \vec{\rho}^{\,k}_{ji} \nabla_j \nabla_j + (\vec{\rho}^{\,kp}_{ji} \cdot \nabla_j \nabla_j + \vec{\rho}^{\,k}_{ji} \cdot \nabla_j \vec{\rho}^{\,p}_{ji} \cdot \nabla_j \nabla_j) \tag{19}$$

$$+ (\vec{\rho}^{\,kpq}_{ji} \cdot \nabla_j \nabla_j + \vec{\rho}^{\,kp}_{ji} \cdot \nabla_j \vec{\rho}^{\,q}_{ji} \cdot \nabla_j \nabla_j + \vec{\rho}^{\,pq}_{ji} \cdot \nabla_j \vec{\rho}^{\,k}_{ji} \cdot \nabla_j \nabla_j$$

$$+ \vec{\rho}^{\,qk}_{ji} \cdot \nabla_j \vec{\rho}^{\,p}_{ji} \cdot \nabla_j \nabla_j + \vec{\rho}^{\,k}_{ji} \cdot \nabla_j \vec{\rho}^{\,p}_{ji} \cdot \nabla_j \vec{\rho}^{\,q}_{ji} \cdot \nabla_j \nabla_j) + \cdots$$

These developments of the operators $D_j - \nabla_j$ and D_{ji} permit one to compute the general planetary perturbations even up to the fourth order if necessary.

We will establish the differential equations for determining the perturbations up to the third order. In our solar system occasions requiring the perturbations of the fourth order will be, probably, very rare. However, in some cases of very sharp commensurabilities of mean motions, the question remains open and further numerical investigations are necessary. Substituting (18) and (19) into (14) and retaining only the typical operators, we have:

$$\frac{d^2}{dt^2} (\vec{r}^{\,k}_i + \vec{r}^{\,kp}_i + \vec{r}^{\,kpq}_i + \cdots) = \tag{20}$$

$$= \mu_i^2 \Big[\vec{r}^{\,k}_i \cdot \nabla_i \nabla_i + (\vec{r}^{\,kp}_i \cdot \nabla_i \nabla_i + \vec{r}^{\,k}_i \cdot \nabla_i \vec{r}^{\,p}_i \cdot \nabla_i \nabla_i)$$

$$+ (\vec{r}^{\,kpq}_i \cdot \nabla_i \nabla_i + \vec{r}^{\,kp}_i \cdot \nabla_i \vec{r}^{\,q}_i \cdot \nabla_i \nabla_i + \vec{r}^{\,pq}_i \cdot \nabla_i \vec{r}^{\,k}_i \cdot \nabla_i \nabla_i$$

$$+ \vec{r}^{\,qk}_i \cdot \nabla_i \vec{r}^{\,p}_i \cdot \nabla_i \nabla_i + \vec{r}^{\,k}_i \cdot \nabla_i \vec{r}^{\,p}_i \cdot \nabla_i \vec{r}^{\,q}_i \cdot \nabla_i \nabla_i) + \cdots\Big] \frac{1}{r_i}$$

$$+ f m_k \Big\{- \Big[\nabla_k + \vec{\rho}^{\,p}_{ki} \cdot \nabla_k \nabla_k + (\vec{\rho}^{\,pq}_{ki} \cdot \nabla_k \nabla_k + \vec{\rho}^{\,p}_{ki} \cdot \nabla_k \vec{\rho}^{\,q}_{ki} \cdot \nabla_k \nabla_k) + \cdots\Big] \frac{1}{\rho_{ki}}$$

$$+ \Big[\nabla_k + \vec{r}^{\,p}_k \cdot \nabla_k \nabla_k + (\vec{r}^{\,pq}_k \cdot \nabla_k \nabla_k + \vec{r}^{\,p}_k \cdot \nabla_k \vec{r}^{\,q}_k \cdot \nabla_k \nabla_k) + \cdots\Big] \frac{1}{r_k}\Big\}$$

$$+ fm_p \left\{ - \left[\vec{\rho}^{\,k}_{pi} \cdot \nabla_p \nabla_p + (\vec{\rho}^{\,kq}_{pi} \cdot \nabla_p \nabla_p + \vec{\rho}^{\,k}_{pi} \cdot \nabla_p \vec{\rho}^{\,q}_{pi} \cdot \nabla_p \nabla_p) + \cdots \right] \frac{1}{\rho_{pi}} \right.$$

$$\left. + \left[\vec{r}^{\,k}_p \cdot \nabla_p \nabla_p + (\vec{r}^{\,qk}_p \cdot \nabla_p \nabla_p + \vec{r}^{\,q}_p \cdot \nabla_p \vec{r}^{\,k}_p \cdot \nabla_p \nabla_p) + \cdots \right] \frac{1}{r_p} \right\}$$

$$+ fm_q \left\{ - \left[(\vec{\rho}^{\,kp}_{qi} \cdot \nabla_q \nabla_q + \vec{\rho}^{\,k}_{qi} \cdot \nabla_q \vec{\rho}^{\,p}_{qi} \cdot \nabla_q \nabla_q) + \cdots \right] \frac{1}{\rho_{qi}} \right.$$

$$\left. + \left[(\vec{r}^{\,kp}_q \cdot \nabla_q \nabla_q + \vec{r}^{\,k}_q \cdot \nabla_q \vec{r}^{\,p}_q \cdot \nabla_q \nabla_q) + \cdots \right] \frac{1}{r_q} \right\} + \cdots$$

Comparing the terms of the same degree in the disturbing masses in the left and right sides of (20), we obtain the basic equations for determining the general perturbations up to the third order:

$$\frac{d^2 \vec{r}^{\,k}_i}{dt^2} = \mu_i^2 \left(\vec{r}^{\,k}_i \cdot \nabla_i \nabla_i \frac{1}{r_i} + \vec{F}^k_i \right), \tag{21}$$

$$\frac{d^2 \vec{r}^{\,kp}_i}{dt^2} = \mu_i^2 \left(\vec{r}^{\,kp}_i \cdot \nabla_i \nabla_i \frac{1}{r_i} + \vec{F}^{kp}_i \right), \tag{22}$$

$$\frac{d^2 \vec{r}^{\,kpq}_i}{dt^2} = \mu_i^2 \left(\vec{r}^{\,kpq}_i \cdot \nabla_i \nabla_i \frac{1}{r_i} + \vec{F}^{kpq}_i \right), \tag{23}$$

where

$$\vec{F}^k_i = \frac{m_k}{1 + m_i} \left(- \nabla_k \frac{1}{\rho_{ki}} + \nabla_k \frac{1}{r_k} \right), \tag{24}$$

$$\vec{F}_i^{kp} = + \vec{r}_i^{\,k}\cdot\nabla_i\ \vec{r}_i^{\,p}\cdot\nabla_i\ \nabla_i\ \frac{1}{r_i} \tag{25}$$

$$+ \frac{m_k}{1+m_i}\left(-\vec{\rho}_{ki}^{\,p}\cdot\nabla_k\ \nabla_k\ \frac{1}{\rho_{ki}} + \vec{r}_k^{\,p}\cdot\nabla_k\ \nabla_k\ \frac{1}{r_k}\right)$$

$$+ \frac{m_p}{1+m_i}\left(-\vec{\rho}_{pi}^{\,k}\cdot\nabla_p\ \nabla_p\ \frac{1}{\rho_{pi}} + \vec{r}_p^{\,k}\cdot\nabla_p\ \nabla_p\ \frac{1}{r_p}\right),$$

and

$$\vec{F}_i^{kpq} = (\vec{r}_i^{\,kp}\cdot\nabla_i\ \vec{r}_i^{\,q}\cdot\nabla_i\ \nabla_i + \vec{r}_i^{\,pq}\cdot\nabla_i\ \vec{r}_i^{\,k}\cdot\nabla_i\ \nabla_i \tag{26}$$

$$+ \vec{r}_i^{\,qk}\cdot\nabla_i\ \vec{r}_i^{\,p}\cdot\nabla_i\ \nabla_i + \vec{r}_i^{\,k}\cdot\nabla_i\ \vec{r}_i^{\,p}\cdot\nabla_i\ \vec{r}_i^{\,q}\cdot\nabla_i\ \nabla_i)\ \frac{1}{r_i}$$

$$+ \frac{m_k}{1+m_i}\left[-(\vec{\rho}_{ki}^{\,pq}\cdot\nabla_k\ \nabla_k + \vec{\rho}_{ki}^{\,p}\cdot\nabla_k\ \vec{\rho}_{ki}^{\,q}\cdot\nabla_k\ \nabla_k)\frac{1}{\rho_{ki}}\right.$$

$$\left. + (\vec{r}_k^{\,pq}\cdot\nabla_k\ \nabla_k + \vec{r}_k^{\,p}\cdot\nabla_k\ \vec{r}_k^{\,q}\cdot\nabla_k\ \nabla_k)\ \frac{1}{r_k}\right]$$

$$+ \frac{m_p}{1+m_i}\left[-(\vec{\rho}_{pi}^{\,qk}\cdot\nabla_p\ \nabla_p + \vec{\rho}_{pi}^{\,q}\cdot\nabla_p\ \vec{\rho}_{pi}^{\,k}\cdot\nabla_p\ \nabla_p)\ \frac{1}{\rho_{pi}}\right.$$

$$\left. + (\vec{r}_p^{\,qk}\cdot\nabla_p\ \nabla_p + \vec{r}_p^{\,q}\cdot\nabla_p\ \vec{r}_p^{\,k}\cdot\nabla_p\ \nabla_p)\ \frac{1}{r_p}\right]$$

$$+ \frac{m_q}{1+m_i}\left[-(\vec{\rho}_{qi}^{\,kp}\cdot\nabla_q\ \nabla_q + \vec{\rho}_{qi}^{\,k}\cdot\nabla_q\ \vec{\rho}_{qi}^{\,p}\cdot\nabla_q\ \nabla_q)\ \frac{1}{\rho_{qi}}\right.$$

$$\left. + (\vec{r}_q^{\,kp}\cdot\nabla_q\ \nabla_q + \vec{r}_q^{\,k}\cdot\nabla_q\ \vec{r}_q^{\,p}\cdot\nabla_q\ \nabla_q)\ \frac{1}{r_q}\right].$$

The equations (21) - (23) can be written in the form

$$\frac{d^2 \vec{r}_i^{\,k}}{dt^2} + \mu_i^2 \left(\frac{I}{r_i^3} - \frac{3\vec{r}_i\, \vec{r}_i}{r_i^5} \right) \cdot \vec{r}_i^{\,k} = \mu_i^2 \vec{F}_i^k . \qquad (27)$$

$$\frac{d^2 \vec{r}_i^{\,kp}}{dt^2} + \mu_i^2 \left(\frac{I}{r_i^3} - \frac{3\vec{r}_i\, \vec{r}_i}{r_i^5} \right) \cdot \vec{r}_i^{\,kp} = \mu_i^2 \vec{F}_i^{kp} \qquad (28)$$

$$\frac{d^2 \vec{r}_i^{\,kpq}}{dt^2} + \mu_i^2 \left(\frac{I}{r_i^3} - \frac{3\vec{r}_i\, \vec{r}_i}{r_i^5} \right) \cdot \vec{r}_i^{\,kpq} = \mu_i^2 \vec{F}_i^{kpq} . \qquad (29)$$

The terms in the right sides of (27) - (29) are the partial gradients of the sums composed of the elementary spherical functions, with the moments equal to the perturbations in $\vec{r}_i$ and $\vec{\rho}_{ki}$ (i, $k = 1, 2, \ldots, n$). Making use of (24) - (26) we obtain

$$F_i^k = \frac{m_k}{1 + m_i} \left(\frac{\vec{\rho}_{ki}}{\rho_{ki}^3} - \frac{\vec{r}_k}{r_k^3} \right); \qquad (30)$$

also the expanded forms of the typical terms in $\vec{F}_i^{kp}$:

$$\vec{r}_i^{\,k} \cdot \nabla_i \; \vec{r}_i^{\,p} \cdot \nabla_i \; \nabla_i \frac{1}{r_i} = \qquad (31)$$

$$= - \frac{15}{r_i^7} \vec{r}_i \; \vec{r}_i \cdot \vec{r}_i^{\,k} \; \vec{r}_i \cdot \vec{r}_i^{\,p}$$

$$+ \frac{3}{r_i^3} \left(\vec{r}_i \cdot \vec{r}_i^{\,k} \; \vec{r}_i^{\,p} + \vec{r}_i \cdot \vec{r}_i^{\,p} \; \vec{r}_i^{\,k} + \vec{r}_i \; \vec{r}_i^{\,k} \cdot \vec{r}_i^{\,p} \right),$$

$$-\frac{mk}{1+m_i}\vec{\rho}^{\,p}_{ki}\cdot\nabla_i\nabla_i\frac{1}{\rho_{ki}} = \frac{m_k}{1+m_i}\left(\frac{I}{\rho^3_{ki}} - \frac{3\vec{\rho}_{ki}\vec{\rho}_{ki}}{\rho^5_{ki}}\right)\cdot\vec{\rho}^{\,p}_{ki}\,, \tag{32}$$

$$+\frac{mk}{1+m_i}\vec{r}^{\,p}_k\cdot\nabla_k\nabla_k\frac{1}{r_k} = -\frac{m_k}{1+m_i}\left(\frac{I}{r^3_k} - \frac{3\vec{r}_k\vec{r}_k}{r^5_k}\right)\cdot\vec{r}^{\,p}_k\,; \tag{33}$$

and the expanded forms of the typical terms in $\vec{F}^{kpq}_i$:

$$\vec{r}^{\,kp}_i\cdot\nabla_i\ \vec{r}^{\,q}_i\cdot\nabla_i\nabla_i\frac{1}{r_i} - \tag{34}$$

$$= -\frac{15}{r^7_i}\vec{r}_i\ \vec{r}_i\cdot\vec{r}^{\,kp}_i\ \vec{r}_i\cdot\vec{r}^{\,q}_i$$

$$+\frac{3}{r^5_i}\left(\vec{r}_i\cdot\vec{r}^{\,kp}_i\ \vec{r}^{\,q}_i + \vec{r}_i\cdot\vec{r}^{\,q}_i\ \vec{r}^{\,kp}_i + \vec{r}_i\ \vec{r}^{\,kp}_i\cdot\vec{r}^{\,q}_i\right);$$

$$\vec{r}^{\,k}_i\cdot\nabla_i\ \vec{r}^{\,p}_i\cdot\nabla_i\ \vec{r}^{\,q}_i\cdot\nabla_i\ \nabla_i\frac{1}{r_i} = \tag{35}$$

$$= +\frac{105}{r^9_i}\vec{r}_i\ \vec{r}_i\cdot\vec{r}^{\,k}_i\ \vec{r}_i\cdot\vec{r}^{\,p}_i\ \vec{r}_i\cdot\vec{r}^{\,q}_i$$

$$-\frac{15}{r^7_i}\left(\vec{r}_i\cdot\vec{r}^{\,p}_i\ \vec{r}_i\cdot\vec{r}^{\,q}_i\ \vec{r}^{\,k}_i + \vec{r}_i\cdot\vec{r}^{\,q}_i\ \vec{r}_i\cdot\vec{r}^{\,k}_i\ \vec{r}^{\,p}_i\right.$$

$$\left.+\vec{r}_i\cdot\vec{r}^{\,k}_i\ \vec{r}_i\cdot\vec{r}^{\,p}_i\ \vec{r}^{\,q}_i + \vec{r}_i\ \vec{r}_i\cdot\vec{r}^{\,k}_i\ \vec{r}^{\,p}_i\cdot\vec{r}^{\,q}_i + \vec{r}_i\ \vec{r}_i\cdot\vec{r}^{\,p}_i\ \vec{r}^{\,k}_i\cdot\vec{r}^{\,q}_i + \vec{r}_i\ \vec{r}_i\cdot\vec{r}^{\,q}_i\ \vec{r}^{\,k}_i\cdot\vec{r}^{\,p}_i\right)$$

$$+\frac{3}{r^5_i}\left(\vec{r}^{\,k}_i\ \vec{r}^{\,p}_i\cdot\vec{r}^{\,q}_i + \vec{r}^{\,p}_i\ \vec{r}^{\,q}_i\cdot\vec{r}^{\,k}_i + \vec{r}^{\,q}_i\ \vec{r}^{\,k}_i\cdot\vec{r}^{\,p}_i\right);$$

$$-\frac{m_k}{1+m_i}\vec{\rho}^{\,pq}_{ki}\cdot\nabla_k\,\nabla_k\,\frac{1}{\rho_{ki}} = \frac{m_k}{1+m_i}\left(\frac{I}{\rho^3_{ki}} - \frac{3\,\vec{\rho}_{ki}\,\vec{\rho}_{ki}}{\rho^5_{ki}}\right)\cdot\vec{\rho}^{\,pq}_{ki}\,, \tag{36}$$

$$-\frac{m_k}{1+m_i}\vec{\rho}^{\,p}_{ki}\cdot\nabla_k\,\vec{\rho}^{\,q}_{ki}\cdot\nabla_k\,\nabla_k\,\frac{1}{\rho_{ki}} = \frac{m_k}{1+m_i}\left[\frac{15}{\rho^7_{ki}}\,\vec{\rho}_{ki}\,\vec{\rho}_{ki}\cdot\vec{\rho}^{\,p}_{ki}\,\vec{\rho}_{ki}\cdot\vec{\rho}^{\,q}_{ki}\right. \tag{37}$$

$$\left.+\frac{3}{\rho^5_{ki}}\left(\vec{\rho}_{ki}\cdot\vec{\rho}^{\,p}_{ki}\,\vec{\rho}^{\,q}_{ki} + \vec{\rho}_{ki}\cdot\vec{\rho}^{\,q}_{ki}\,\vec{\rho}^{\,p}_{ki} + \vec{\rho}_{ki}\,\vec{\rho}^{\,p}_{ki}\cdot\vec{\rho}^{\,q}_{ki}\right)\right],$$

$$\frac{m_k}{1+m_i}\vec{r}^{\,pq}_{k}\cdot\nabla_k\,\nabla_k\,\frac{1}{r_k} = -\frac{m_k}{1+m_i}\left(\frac{I}{r^3_k} - \frac{3\,\vec{r}_k\,\vec{r}_k}{r^5_k}\right)\cdot\vec{r}^{\,pq}_{k}\,, \tag{38}$$

$$\frac{m_k}{1+m_i}\vec{r}^{\,q}_{k}\cdot\nabla_k\,\vec{r}^{\,q}_{k}\cdot\nabla_k\,\nabla_k\,\frac{1}{r_k} = \tag{39}$$

$$= -\frac{m_k}{1+m_i}\left[\frac{15}{r^7_k}\,\vec{r}_k\,\vec{r}_k\cdot\vec{r}^{\,p}_{k}\,\vec{r}_k\cdot\vec{r}^{\,q}_{k}\right.$$

$$\left.-\frac{3}{r^5_k}\left(\vec{r}_k\cdot\vec{r}^{\,p}_{k}\,\vec{r}^{\,q}_{k} + \vec{r}_k\cdot\vec{r}^{\,q}_{k}\,\vec{r}^{\,p}_{k} + \vec{r}_k\,\vec{r}^{\,p}_{k}\cdot\vec{r}^{\,q}_{k}\right)\right].$$

In a similar way the expressions for $\vec{F}^{\,kpqs}_{i}$ and for its typical terms can be formed. The process can be continued as far as necessary, until the perturbations become negligible. The restriction that indices k, p, q,... be different now can be removed.

Integration Procedure

The typical differential equation which appears in the problem of the general perturbations in the position vectors has the form

$$\frac{d^2\vec{x}_i}{dt^2} + \mu_i^2\left(\frac{\vec{x}_i}{r_i^3} - \frac{3\,\vec{x}_i\cdot\vec{r}_i\,\vec{r}_i}{r_i^5}\right) = \mu_i^2\,\vec{F}_i, \tag{40}$$

where

$$\vec{x}_i = \vec{r}_i^{\,k},\quad \vec{r}_i^{\,kp},\quad \vec{r}_i^{\,kpq},\ \ldots$$

$$\vec{F}_i = \vec{F}_i^{\,k},\quad \vec{F}_i^{\,kp},\quad \vec{F}_i^{\,kpq},\ \ldots$$

and $\vec{F}_i$ is a series with the periodic, secular and mixed terms. Equation (40) is the variational equation of the two body problems.

In order to reduce the solutions of (40) to quadratures we shall make use of the substitution

$$\vec{x}_i = (s_i + 2w_i)\,\vec{r}_i - \vec{v}_i\int(2s_i + 3w_i)\,dt + \zeta_i\,\vec{R}_i. \tag{41}$$

This differs from the substitutions used by the author [1963] previously. The substitution (41) was chosen because of a simpler form to which (40) is reduced thereby as compared with the earlier exposition. It is unnecessary to retain the index i in the further exposition: we can now omit it without loss of clarity.

It follows from (41) that

$$\frac{d\vec{x}}{dt} = \left\{\frac{ds}{dt} + 2\,\frac{dw}{dt} + \frac{\mu^2}{r^3}\int(2s + 3w)\,dt\right\}\vec{r} - (s + w)\,\vec{v} + \frac{d\zeta}{dt}\,\vec{R}, \tag{42}$$

$$\frac{d^2\vec{x}}{dt^2} = \left\{\frac{d^2s}{dt^2} + 2\,\frac{d^2w}{dt^2} - \frac{3\mu^2}{r^4}\,\frac{dr}{dt}\int(2s + 3w)\,dt \right. \tag{43}$$

$$\left. + \frac{\mu^2}{r^3}\,(3s + 4w)\right\}\vec{r} + \left\{\frac{dw}{dt} + \frac{\mu^2}{r^3}\int(2s + 3w)\,dt\right\}\vec{v}$$

$$+ \frac{d^2\zeta}{dt^2}\,\vec{R}.$$

Substituting (41) and (43) into (40), we obtain the vectorial differential equation

$$\left(\frac{d^2 s}{dt^2} + \frac{\mu^2}{r^3}\, s + 2\,\frac{d^2 w}{dt^2}\right)\vec{r} + \frac{dw}{dt}\,\vec{v} + \left(\frac{d^2 \zeta}{dt^2} + \frac{\mu^2}{r^3}\,\zeta\right)\vec{R} = \mu^2 \vec{F}, \tag{44}$$

which can be integrated by quadratures. Forming the dot products of (44) and

$$\vec{v} \times \vec{R}, \quad \vec{R} \times \vec{r}, \quad \vec{R}$$

and taking

$$\vec{r} \cdot \vec{v} \times \vec{R} = \vec{v} \cdot \vec{R} \times \vec{r} = \mu \sqrt{a(1 - e^2)}$$

$$\vec{r} \cdot \vec{v} = r\,\frac{dr}{dt}$$

into account, we have

$$\frac{d^2 s}{dt^2} + \frac{\mu^2}{r^3}\, s + 2\,\frac{d^2 w}{dt^2} = \frac{na}{\sqrt{1 - e^2}}\;\vec{F} \cdot \vec{v} \times \vec{R}, \tag{45}$$

$$\frac{dw}{dt} = \frac{na}{\sqrt{1 - e^2}}\,\vec{F} \cdot \vec{R} \times \vec{r}, \tag{46}$$

$$\frac{d^2 \zeta}{dt^2} + \frac{\mu^2}{r^3}\,\zeta = \mu^2\,\vec{F} \cdot \vec{R}. \tag{47}$$

We have from (46)

$$w = K_3 + B, \tag{48}$$

where we put

$$B = \int \frac{na}{\sqrt{1-e^2}} \vec{F}\cdot\vec{R}\times\vec{\bar{r}}\, dt; \tag{49}$$

K_3 is the additive constant of integration. The integration of series in (49) is performed in a formal manner.

The equation (45) can now be integrated using Hill's procedure [1874]. We obtain:

$$s = K_1 \frac{r}{a} \cos f + K_2 \frac{r}{a} \sin f + A, \tag{50}$$

where K_1 and K_2 are constants of integration and we put

$$A = \int \frac{\left(\vec{F}\cdot\vec{v}\times\vec{R}\right)\left(\vec{R}\cdot\vec{r}\times\vec{\bar{r}}\right)}{a\left(1-e^2\right)}\, dt - \int \frac{2}{m^2\sqrt{1-e^2}} \frac{d^2 w}{dt^2}\left(\vec{R}\cdot\vec{r}\times\vec{\bar{r}}\right) dt. \tag{51}$$

The vector $\vec{\bar{r}}$ is considered as a temporary constant and it is replaced by $\vec{r}$ after the integration is completed. The integrand is a trigonometrical series in the mean anomalies of planets and the auxiliary mean anomaly $\bar{\ell}$ associated with $\vec{\bar{r}}$, and it can also contain the purely secular and the mixed terms. After the integration is performed $\bar{\ell}$ has to be replaced by ℓ. Integrating the second integral in (51) by parts and replacing $\vec{r}\times\vec{\bar{r}}$ by zero when it appears outside the integral sign, we obtain

$$\int \frac{d^2 w}{dt^2} \vec{R}\cdot\vec{r}\times\vec{\bar{r}}\, dt = -\int \frac{dw}{dt} \vec{R}\cdot\vec{v}\times\vec{\bar{r}}\, dt \tag{52}$$

From (46), (51) and (52) we deduce:

$$A = \int \frac{\vec{R}\cdot\vec{r}\times\vec{\bar{r}}}{a\left(1-e^2\right)}\left(\vec{F}\cdot\vec{v}\times\vec{R}\right) dt \tag{53}$$

$$+ \int \frac{2\vec{R}\cdot\vec{v}\times\vec{\bar{r}}}{a\left(1-e^2\right)}\left(\vec{F}\cdot\vec{R}\times\vec{r}\right) dt .$$

As in (49), the integration is performed in a formal manner.

In a similar way we obtain from (47)

$$\zeta = K_5 \frac{r}{a} \cos f + K_6 \frac{r}{a} \sin f + Z , \tag{54}$$

where

$$Z = \int \frac{n a}{\sqrt{1-e^2}} \left(\vec{F}\cdot\vec{R}\right)\left(\vec{R}\cdot\vec{r}\times\vec{\bar{r}}\right) dt \tag{55}$$

and K_5, K_6 are constants of integration. From (48) and (50), and considering

$$\int n \frac{r}{a} \cos f \, dt = -\frac{3}{2} e\, n t + \frac{\sqrt{1-e^2}}{2} \frac{r}{a} \sin f + \frac{1}{2\sqrt{1-e^2}} \frac{r^2}{a^2} \sin f ,$$

$$\int n \frac{r}{a} \sin f \, dt = -\sqrt{1-e^2}\, \frac{r^2}{a^2} \left(\cos f + \frac{1}{2} e \cos^2 f\right) ,$$

we obtain

$$\int (2s + 3w)\, dt = \int (2A + 3B)\, dt \tag{56}$$

$$+ \frac{3}{n} \left(K_3 - e\, K_1\right) nt + \frac{K_1}{n\sqrt{1-e^2}} \frac{r^2}{a^2} \left(2 \sin f + \frac{1}{2} e \sin 2f\right)$$

$$- \frac{K_2 \sqrt{1-e^2}}{n} \frac{r^2}{a^2} \left(\frac{1}{2} e + 2 \cos f + \frac{1}{2} e \cos 2f\right) + K_4 ,$$

where K_4 is the additive constant of integration.

The forms (30) - (39) of the disturbing terms in the right sides of the variational equations require the decomposition of $\vec{r}_i^{\,k}$, $\vec{r}_i^{\,kp}$, $\vec{r}_i^{\,kpq}$, ... in the moving frame $\vec{r}_i$, $\vec{R}_i \times \vec{r}_i$, $\vec{R}_i$ rather than in the frame $\vec{r}_i$, $\vec{v}_i$, $\vec{R}_i$. Setting

$$\vec{x} = \xi\, \vec{r}^{\,0} + \eta\, \vec{R} \times \vec{r}^{\,0} + \zeta\, \vec{R} \tag{57}$$

and taking

$$\vec{r}^{\,0} \cdot \vec{v} = \frac{dr}{dt} = \frac{na\; e \sin f}{\sqrt{1-e^2}} ,$$

$$\vec{v} \cdot \vec{R} \times \vec{r}^{\,0} = \frac{na^2 \sqrt{1-e^2}}{r}$$

into consideration, we deduce from (41)

$$\xi = (s + 2w)\, r - \frac{na\; e \sin f}{\sqrt{1-e^2}} \int (2s + 3w)\, dt , \tag{58}$$

$$\eta = -\frac{na^2\sqrt{1-e^2}}{r}\int(2s + 3w)\,dt. \qquad (59)$$

The decomposition (57) was suggested by Popović [1960, 1961] and independently by the author [1964]. In Popović's work, the perturbations of the first and of the second order in ξ, η, ζ are determ d.

The disturbing vector $\vec{F}$ can also be decomposed along $\vec{r}$, $\vec{R}\times\vec{r}$, and $\vec{R}$. Perhaps from the computational standpoint this decomposition is the simplest one; it appears in several theories of the general planetary perturbations, either directly or as an intermediary step. Let us put

$$S = \vec{F}\times\vec{r}, \quad T = \vec{F}\cdot\vec{R}\times\vec{r},$$

From

$$\vec{v} = \frac{n}{r}\frac{dr}{d\ell}\vec{r} + \frac{na^2\sqrt{1-e^2}}{r^2}\vec{R}\times\vec{r},$$

we deduce

$$\vec{F}\cdot\vec{v}\times\vec{R} = \frac{na^2\sqrt{1-e^2}}{r^2}S - \frac{n}{r}\frac{dr}{d\ell}T.$$

Substituting this value into (53), we obtain

$$A = \int\left[\vec{r}\left(\frac{na^2\sqrt{1-e^2}}{r^2}S - \frac{n}{r}\frac{dr}{d\ell}T\right) + 2\vec{v}\,T\right]\cdot\frac{\vec{r}\times\vec{R}}{a\left(1-e^2\right)}dt;$$

and replacing $\vec{v}$ by its decomposition as given above we have, after some easy vectorial transformations,

$$A = \int(MS + NT)\,dt, \qquad (53')$$

where we put

$$M = \frac{na}{\sqrt{1-e^2}} \left(\frac{a}{r}\right)^2 \frac{\bar r}{a} \frac{r}{a} \sin \left(\bar f - f\right) ,$$

$$N = \frac{a}{r} \cdot \frac{na}{\left(1-e^2\right)^{3/2}} \left[\sqrt{1-e^2} \left(\frac{d}{d\ell} \frac{r}{a}\right) \frac{\bar r}{a} \frac{r}{a} \sin \left(\bar f - f\right) - \frac{2\, a \left(1-e^2\right)}{r} \frac{\bar r}{a} \frac{r}{a} \cos \left(\bar f - f\right)\right];$$

M is a sine series in ℓ and $\bar\ell$, and N is a cosine series in the same arguments. In order to obtain these series we have to obtain the development of

$$\frac{r}{a}, \frac{a}{r}, \frac{r}{a} \cos f, \text{ and } \frac{r}{a} \sin f.$$

The equation (49) can be written as

$$B = \int \frac{na\, T}{\sqrt{1-e^2}}\, dt . \tag{49'}$$

S

The computation of integrands in (49'), (53') and (55) requires also the development of some other expressions: for example, of the odd powers of $1/\rho_{ji}$; of powers of a_k/r_k; of the scalar products $\vec r_i \cdot \vec r_j$ and $\vec r_j \cdot \vec R_i$; and the triple products $\vec R_i \cdot \vec r_i \times \vec r_j$. It seems that the simplest and easiest way to obtain all these developments is by means of the single and double harmonic analyses. The formulas

$$\vec r_i = \vec A_i \left(\cos \epsilon_i - e_i\right) + \vec B_i \sin \epsilon_i ,$$

$$\epsilon_i - e_i \sin \epsilon_i = \ell_i ,$$

$$\vec{A}_i = a_i \vec{P}_i, \qquad \vec{B}_i = a_i \sqrt{1-e_i^2}\, \vec{Q}_i$$

serve as a start. The representation of Δ^2_{ji} in terms of the vectorial elements is very useful in performing the double harmonic analysis.

Determination of the Constants of Integration

Each perturbation

$$\vec{r}_i^{\,k}, \ \vec{r}_i^{\,kp}, \ \vec{r}_i^{\,kpq}, \ \ldots$$

introduces six constants of integration. We shall designate them by

$$K^k_{ji}, \ K^{kp}_{ji}, \ K^{kpq}_{ji}, \ \ldots \qquad i,k,p,q, \ \ldots = 1,2, \ \ldots, n$$

$$j = 1,2, \ \ldots 6$$

correspondingly. Consequently, the problem of determining the general perturbations by developing them into power series with respect to the masses introduces an infinite set of constants of integration. Of course, with the increasing number of the upper indices these constants decrease rapidly.

In the final expressions for the perturbations, the constants are combined to form a set of only 6n independent constants; but this is not seen explicitely in a numerical planetary theory. At each step the constants must be determined separately from some additional conditions imposed on the elements or from the initial position and velocity vectors. Two types of elements are being commonly used: the mean elements and the elements osculating at the initial moment of time. In the case of the mean elements the perturbations of the true longitude in the orbital plane, as defined by these elements, shall not contain the constant term, the purely secular term and the terms with periods equal to the period of revolution of the planet. The perturbations in the "third coordinate," normal to the orbital plane, shall not contain the terms with periods equal to the revolution of the planet. The values of the mean

elements, however, are not unique. They depend upon the choice of the eccentric, true, or mean anomaly as the basic independent variable to be used in developing the perturbations of a given planet.

Hansen [1857] made use of the eccentric anomaly in his theory of minor planets. Hill [1874] preferred the true anomaly. In both choices we gain speed of convergence of series giving the perturbations of the first order, in comparison to the choice of the mean anomaly. However, the road to computation of the higher order perturbations of the whole planetary system by either of these two choices will be blocked so effectively that all the gains in the first approximation appear to be negated by the difficulties encountered in computing these higher order perturbations. For this reason the use of the undisturbed mean anomalies $\ell_1, \ell_2, \ldots, \ell_n$ and, consequently, of the universal variable, time, is highly recommended in the planetary theories. This is done already by Hansen and by Hill in their theories of Jupiter and Saturn. In connection with this statement we say that the elements are mean if there are no terms of the form

$$K_0,\ K_1 t,\ K^{(c)} \cos \ell,\ K^{(s)} \sin \ell$$

in the perturbations of the true longitudes with respect to the undisturbed orbit planes, and there are no terms of the form

$$K^{(c)} \cos \ell,\ K^{(s)} \sin \ell$$

in the "third coordinates" ζ.

We express the perturbations in the true longitude in terms of the perturbations along $\overline{r}^0$, $\overline{R} \times \overline{r}^0$, and $\overline{R}$. Taking into account the equations

$$\frac{\partial \overline{r}}{\partial r} = \overline{r}^0, \quad \frac{\partial \overline{r}}{\partial \lambda} = r\, \overline{R} \times \overline{r}^0,$$

$$\frac{\partial^2 \vec{r}}{\partial r^2} = 0, \quad \frac{\partial^2 \vec{r}}{\partial r\, \partial \lambda} = \vec{R}\times\vec{r}^{\,0}, \quad \frac{\partial^2 \vec{r}}{\partial \lambda^2} = - r\, \vec{r}^{\,0}$$

$$\frac{\partial^3 \vec{r}}{\partial r^3} = 0, \quad \frac{\partial^3 \vec{r}}{\partial r^2\, \partial \lambda} = 0, \quad \frac{\partial^3 \vec{r}}{\partial r\, \partial \lambda^2} = - \vec{r}^{\,0}, \quad \frac{\partial^3 \vec{r}}{\partial \lambda^3} = - r\, \vec{R}\times\vec{r}^{\,0},$$

We can write

$$\delta\vec{r} = \xi\vec{r}^{\,0} + \eta\, \vec{R}\times\vec{r}^{\,0} + \zeta\vec{R} =$$

$$= \left(\delta r - \frac{1}{2} r\, \delta\lambda^2 - \frac{1}{2}\delta r\, \delta\lambda^2 + \ldots\right)\ \vec{r}^{\,0}$$

$$+ \left(r\, \delta\lambda + \delta r\, \delta\lambda - \frac{1}{6} r\, \delta\lambda^3 + \ldots\right)\ \vec{R}\times\vec{r}^{\,0} + \zeta\vec{R},$$

or

$$\xi = \delta r - \frac{1}{2} r\, \delta\lambda^2 - \frac{1}{2}\delta r\, \delta\lambda^2 + \ldots, \tag{60}$$

$$\eta = r\, \delta\lambda + \delta r\, \delta\lambda - \frac{1}{6} r\, \delta\lambda^3 + \ldots. \tag{61}$$

Solving these two last equations with respect to δr, $\delta\lambda$, we have

$$\delta r = \xi + \frac{\eta^2}{2r} - \frac{\xi\eta^2}{2r^2} + \ldots, \tag{62}$$

$$\delta\lambda = \frac{\eta}{r} - \frac{\xi\eta}{r^2} - \frac{1}{3}\frac{\eta^3}{r^3} + \frac{\xi^2\eta}{r^3} + \ldots. \tag{63}$$

Substituting

$$\xi = \left(\xi^k + \xi^p + \xi^q\right) + \left(\xi^{kp} + \xi^{pq} + \xi^{qk}\right) + \xi^{kpq} + \ldots ,$$

$$\eta = \left(\eta^k + \eta^p + \eta^q\right) + \left(\eta^{kp} + \eta^{pq} + \eta^{qk}\right) + \eta^{kpq} + \ldots ,$$

$$\delta\lambda = \lambda^k + \lambda^{kp} + \lambda^{kpq} + \ldots ,$$

we obtain into (63)

$$\lambda^k = \frac{\eta^k}{r}, \tag{64}$$

$$\lambda^{kp} = \frac{\eta^{kp}}{r} - \frac{\xi^p \eta^k + \xi^k \eta^p}{r^2}, \tag{65}$$

$$\lambda^{kpq} = \frac{\eta^{kpq}}{r} - \frac{1}{r^2}\left(\xi^k \eta^{pq} + \xi^p \eta^{qk} + \xi^q \eta^{kp}\right) \tag{66}$$

$$+ \frac{2}{r^3}\left(\xi^k \xi^p \eta^q + \xi^p \xi^q \eta^k + \xi^q \xi^k \eta^p\right)$$

$$- \frac{1}{r^3}\left(\xi^{kp} \eta^q + \xi^{pq} \eta^k + \xi^{qk} \eta^p\right)$$

$$- \frac{2}{r^3} \eta^k \eta^p \eta^q ,$$

. . . .

Putting:

$$n_i \sqrt{1-e_i^2}\, W_i^k = - \frac{n_i\, a_i^2 \sqrt{1-e_i^2}}{r_i^2} \int \left(2A_i^k + 3B_i^k\right) dt, \tag{67}$$

$$n_i \sqrt{1-e_i^2}\, W_i^{kp} = - \frac{n_i\, a_i^2 \sqrt{1-e_i^2}}{r_i^2} \int \left(2A_i^{kp} + 3B_i^{kp}\right) dt - \frac{\xi_i^p\, \eta_i^k + \xi_i^k\, \eta_i^p}{r_i^2}. \tag{68}$$

$$\begin{aligned} n_i \sqrt{1-e_i^2}\, W_i^{kpq} = &- \frac{n_i\, a_i^2 \sqrt{1-e_i^2}}{r_i^2} \int \left(2A_i^{kpq} + 3B_i^{kpq}\right) dt \\ &- \frac{1}{r_i^2} \left(\xi_i^k\, \eta_i^{pq} + \xi_i^p\, \eta_i^{qk} + \xi_i^q\, \eta_i^{kp}\right) \\ &+ \frac{2}{r_i} \left(\xi_i^k\, \xi_i^p\, \eta_i^q + \xi_i^p\, \xi_i^q\, \eta_i^k + \xi_i^q\, \xi_i^k\, \eta_i^p\right) \\ &- \frac{1}{r_i^3} \left(\xi_i^{kp}\, \eta_i^q + \xi_i^{pq}\, \eta_i^k + \xi_i^{qk}\, \eta_i^p\right) \\ &- \frac{2}{r_i^3}\, \eta_i^k\, \eta_i^p\, \eta_i^q, \end{aligned} \tag{69}$$

And substituting these values into (64) - (66) and taking (59) into consideration we obtain

$$\frac{\lambda_i^k}{n_i\sqrt{1-e_i^2}} = \frac{3}{n_i}\left(\frac{a_i}{r_i}\right)^2 \left(-K_{3i}^k + e_i K_{1i}^k\right) \dot{n}_i t \tag{70}$$

$$-\frac{K_{1i}^k}{n_i\sqrt{1-e_i^2}}\left(2\sin f_i + \frac{1}{2}e_i \sin 2f_i\right)$$

$$+\frac{K_{2i}^k\sqrt{1-e_i^2}}{n_i}\left(2\cos f_i + \frac{1}{2}e_i + \frac{1}{2}e_i\cos 2f_i\right)$$

$$+\frac{a_i^2}{r_i^2}K_{4i}^k + W_i^k$$

$$\frac{\lambda_i^{kp}}{n_i\sqrt{1-e_i^2}} = \frac{3}{n_i}\left(\frac{a_i}{r_i}\right)^2 \left(-K_{3i}^{kp} + e_i K_{1i}^{kp}\right) n_i t$$

$$-\frac{K_{1i}^{kp}}{n_i\sqrt{1-e_i^2}}\left(2\sin f_i + \frac{1}{2}e_i \sin 2f_i\right)$$

$$+\frac{K_{2i}^{kp}\sqrt{1-e_i^2}}{n_i}\left(2\cos f_i + \frac{1}{2}e_i + \frac{1}{2}e_i\cos 2f_i\right)$$

$$+\frac{a_i^2}{r_i^2}K_{4i}^{kp} + W_i^{kp} \tag{71}$$

$$\frac{\lambda_i^{kpq}}{n_i\sqrt{1-e_i^2}} = \frac{3}{n_i}\left(\frac{a_i}{r_i}\right)^2 \left(-K_{3i}^{kpq} + e_i K_{1i}^{kpq}\right) n_i t$$

$$-\frac{K_{1i}^{kpq}}{n_i\sqrt{1-e_i^2}}\left(2\sin f_i + \frac{1}{2}e_i \sin 2f_i\right)$$

$$+\frac{K_{2i}^{kpq}\sqrt{1-e_i^2}}{n_i}\left(2\cos f_i + \frac{1}{2}e_i + \frac{1}{2}e_i\cos 2f_i\right)$$

$$+\frac{a_i^2}{r_i^2}K_{4i}^{kpq} + W_i^{kpq} \tag{72}$$

We shall make use of the developments

$$\left(\frac{r_i}{a_i}\right)^p \cos qf_i = \frac{1}{2}C_{0i}^{pq} + C_{1i}^{pq}\cos \ell_i + C_{2i}^{pq}\cos 2\ell_i + \ldots ,$$

$$\left(\frac{r_i}{a_i}\right)^p \sin qf_i = S_{1i}^{pq}\sin \ell_i + S_{2i}^{pq}\sin 2\ell_i + \ldots .$$

The coefficients in these developments are computed either by some analytical classic procedure or by means of a harmonic analysis if the eccentricity is not too small.

The terms of the form

$$K_{0i},\ K_i t,\ K_i^{(c)}\cos \ell,\ K_i^{(s)}\sin \ell \tag{73}$$

must be absent in the developments of (70) - (71). We have, keeping only the substantial terms,

$$W_i^k = \alpha_{0i}^k + \beta_{0i}^k n_i t + \alpha_{1i}^k \cos \ell_i + \beta_{1i}^k \sin \ell_i + \ldots , \tag{74}$$

$$W_i^{kp} = \alpha_{0i}^{kp} + \beta_{0i}^{kp} n_i t + \alpha_{1i}^{kp} \cos \ell_i + \beta_{1i}^{kp} \sin \ell_i + \dots , \qquad (75)$$

$$W_i^{kpq} = \alpha_{0i}^{kpq} + \beta_{0i}^{kpq} n_i t + \alpha_{1i}^{kpq} \cos \ell_i + \beta_{1i}^{kpq} \sin \ell_i + \dots . \qquad (76)$$

In the process of computing the coefficients α and β, the machine rejects automatically all the useless terms, unless we decide to obtain a complete development of the perturbations in the true longitude. The conditions for the absence of the terms (73) in the right sides of (70)-(71) lead to the equations:

$$+ \frac{K_2 \sqrt{1-e^2}}{n} \left(C_0^{0,1} + \frac{1}{2} e + \frac{1}{4} C_0^{0,2} \right) + \frac{1}{2} K_4 C_0^{-2,0} + \alpha_0 = 0, \qquad (77)$$

$$- \frac{K_1}{n \sqrt{1-e^2}} \left(2 S_1^{0,1} + \frac{1}{2} e \, S_1^{0,2} \right) + \beta_1 = 0, \qquad (78)$$

$$+ \frac{K_2 \sqrt{1-e^2}}{n} \left(2 C_1^{0,1} + \frac{1}{2} e \, C_1^{0,2} \right) + C_1^{-2,0} K_4 + \alpha_1 = 0 \qquad (79)$$

$$+ \frac{3}{2n} C_0^{-2,0} \left(- K_3 + e K_1 \right) + \beta_0 = 0. \qquad (80)$$

Separating in Z_i^k, Z_i^{kp}, Z_i^{kpq}, ... the terms with the argument ℓ, we have

$$Z_i^k = c_{1i}^k \cos \ell + s_{1i}^k \sin \ell + \dots ,$$

$$Z_i^{kp} = c_{1i}^{kp} \cos \ell + s_{1i}^{kp} \sin \ell + \dots ,$$

$$Z_i^{kpq} = c_{1i}^{kpq} \cos \ell + s_{1i}^{kpq} \sin \ell + \ldots ;$$

$$(i, k, p, q, \ldots = 1, 2, \ldots, n)$$

and the conditions for the absence of terms with the argument ℓ in ζ^k, ζ^{kp}, $\zeta^{kpq}, \ldots$ lead to

$$K_5 C_1^{1,1} + c_1 = 0, \tag{81}$$

$$K_6 S_1^{1,1} + s_1 = 0. \tag{82}$$

The lower index i and the upper indices k, p, q, ... are omitted in (77) - (82). Evidently, this is not causing any ambiguity.

Conclusion

The results given in this article represent the extension and completion of the results given in the author's previous articles on this subject. The theory given here can also be considered as a modification and generalization of Hill's planetary theory, with the latter's inconveniences removed. The interdependent constants of integration peculiar to Hill's theory do not appear in the present exposition. The solution is given in the form which permits us to write immediately the differential equation for the general perturbations proportional to any prescribed product of masses. Moreover, the vectorial formalism permits us to penetrate into the structure of higher orders effects without great difficulty. Programming is also facilitated by the repetition of the homogeneous operations.

The formula (57) permits one to obtain easily the decomposition of $\vec{r}_i^{\alpha}$, $\vec{r}_i^{\alpha\beta}$, ... along the axes of the inertial systems, if it is considered as necessary.

On the basis of experience obtained at Goddard Space Flight Center one might expect that computing the perturbations of a given order for one planet will require only a few minutes. Considering the simplicity of the methods for the general perturbations in the position vectors, one is inclined to believe that such methods will consitiute one of the principal approaches to the problem in the not too distant future.

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